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Finite Groups of Automorphisms of Infinite Groups II

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This paper studies Aut A in the case in which A is an infinite abelian group and Aut A is finite. Information is obtained about the structure of Aut A including the description of a large normal subgroup. Aut A is completely characterized when it is abelian. Its order is determined when it is dihedral or generalized quaternion.

1. INTRODUCTION

In a series of papers culminating in [9] Hallett and Hirsch have completely classified the finite groups that occur as the full automorphism groups of infinite torsion-free abelian groups. (see [4, Chap. XVI; 5–9] and also [10]). They are the subdirect products of cyclic groups of order 2, 4, or 6, the quaternion group of order 8, the dicyclic group of order 12, and the special linear group $SL(2, 3)$ of order 24 that satisfy the following condition: if they contain an element of order 12, then they contain at least one element of order 2 that is not a sixth power. It seems that it would be difficult to extend this type of classification to a class of groups much larger than the torsion-free abelian groups. However, by restricting our attention to certain classes of finite groups we may be able to determine which members of those classes can arise as the automorphism groups of infinite groups. For example, de Vries and de Miranda [12] have determined the groups of order 8 or less that can be automorphism groups. In [1] we proved the following.

THEOREM 1.1. *There is an infinite non-abelian group G with* Aut $G = A$ *for*

(i) $A = S_n$ *if and only if* $n = 4$,

(ii) $A = D_n$, *the dihedral group of order* $2n$, *if and only if* $n = 6$,

and there is no infinite non-abelian group G with Aut $G = A$ *for*

(iii) *A cyclic,*

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(iv) *A a finite group with $A = A'$, A/ZA a non-abelian simple group, and ZA a cyclic group,*

(v) *$A = A_n$, the alternating group,*

(vi) *$A = Q_n$, the generalized quaternion group of order 2^n, $n \geqslant 3$.*

In [2, 3] it was shown that an elementary abelian p-group may occur as the automorphism group of an infinite non-abelian group for any prime p (although there are certain restrictions on the ranks). In this paper we study various classes of finite groups and determine which members of these classes can be the automorphism groups of infinite abelian groups. Besides obtaining information about the structure of $\operatorname{Aut} A$ including the description of a large normal subgroup we will prove the following theorem.

THEOREM 1.2. *There is an infinite abelian group A with $\operatorname{Aut} A = G$ for*

(i) *G a finite abelian group if and only if G is of even order and is a direct product of cyclic groups of orders 2, 3, and 4 with the property that if G has an element of order 12 it also has an element of order 2 that is not a sixth power.*

(ii) *$G = D_n$, the dihedral group of order $2n$, if and only if $n = 1$, 2, or 6.*

(iii) *$G = Q_n$, the generalized quaternion group of order 2^n, $n \geqslant 3$, if and only if $n = 3$.*

There is never an infinite abelian group A with $\operatorname{Aut} A = G$ where

(iv) *there is no element of order 2 in ZG.*

(v) *G is finite, G/ZG is a non-abelian simple group, and ZG is cyclic.*

Notation.

C_n: the cyclic group of order n,
D_n: the dihedral group of order $2n$,
Q_n: the generalized quaternion group of order 2^n, $n \geqslant 3$,
ZG: the center of the group G,
$A \lhd B$: the split extension of the group A by the group B,
$-I_A$: the automorphism of the abelian group A that sends x to $-x$.

2. STRUCTURE OF A AND $\operatorname{Aut} A$

Suppose that A is an infinite abelian group and that $\operatorname{Aut} A$ is finite. A theorem of Nagrebeckiĭ [11] asserts that in an infinite group with finitely many automorphisms the elements of finite order form a finite subgroup.

Hence, $A = F \oplus T$ where T is finite and F is torsion-free and non-trivial. Since F has elements that are not of order 2, the automorphism $x \to -x$ has order 2. It is clear that this automorphism is also central in $\operatorname{Aut} A$. Thus, the order of the center of $\operatorname{Aut} A$ is divisible by 2. This immediately eliminates many possibilities for $\operatorname{Aut} A$. For example, $\operatorname{Aut} A$ can never be the symmetric group S_n for $n \geqslant 3$, the alternating group A_n for $n \geqslant 3$, a finite simple group, or the dihedral group D_n of order $2n$ where n is odd. This is in sharp contrast to finite abelian groups. For example, if $C_2^{(n)}$ is an elementary abelian 2-group of order 2^n, then $\operatorname{Aut} C_2^{(n)} \cong PSL(n, 2)$ which is centerless and indeed simple for $n \geqslant 3$.

We will now investigate the structure of $\operatorname{Aut} A$. Since $A = F \oplus T$ we may represent $\operatorname{Aut} A$ as the group of matrices

$$\begin{pmatrix} \alpha & \theta \\ 0 & \beta \end{pmatrix},$$

where $\alpha \in \operatorname{Aut} F$, $\beta \in \operatorname{Aut} T$, and $\theta \in \operatorname{Hom}(F, T)$. (Here we are assuming, of course, that automorphisms act on the right.) The mapping

$$\begin{pmatrix} \alpha & \theta \\ 0 & \beta \end{pmatrix} \to (\alpha, \beta)$$

determines a homomorphism from $\operatorname{Aut} A$ onto $\operatorname{Aut} F \times \operatorname{Aut} T$ giving rise to the split exact sequence

$$\operatorname{Hom}(F, T) \rightarrowtail \operatorname{Aut} A \twoheadrightarrow \operatorname{Aut} F \times \operatorname{Aut} T. \tag{2.1}$$

It remains to determine $\operatorname{Hom}(F, T)$ and the action of $\operatorname{Aut} F$ and $\operatorname{Aut} T$ on $\operatorname{Hom}(F, T)$. We begin with the following result.

LEMMA 2.1. *Suppose that* $\operatorname{Aut}(F \oplus T)$ *is finite. If* T_p *is the p-primary component of* T, *then* $T_p = 0$ *if and only if* $\operatorname{Hom}(F, T_p) = 0$.

Proof. If $T_p = 0$, then clearly $\operatorname{Hom}(F, T_p) = 0$. Suppose that $\operatorname{Hom}(F, T_p) = 0$. Since F/pF is an elementary abelian p-group, it follows that $F = pF$. Hence, the mapping $x \to px$ is an automorphism of F of infinite order that extends to $F \oplus T$. This contradiction completes the proof.

Since $T = \oplus_p T_p$, we have $\operatorname{Hom}(F, T) \simeq \oplus_p \operatorname{Hom}(F, T_p)$. Let p^n be the order of T_p. Since F is torsion-free, $F/p^n F$ is a direct sum of cyclic groups of order p^n and the number of summands is $d(p)$, the dimension of F/pF over the integers modulo p. Since $\operatorname{Hom}(C_{p^n}, T_p) \simeq T_p$, we have

$$\operatorname{Hom}(F, T_p) \simeq \operatorname{Hom}(F/p^n F, T_p) \simeq \bigoplus_{i=1}^{d(p)} T_p.$$

Hence,

$$\operatorname{Hom}(F, T) \simeq \bigoplus_p \operatorname{Hom}(F, T_p) \simeq \bigoplus_p \bigoplus_{i=1}^{d(p)} T_p = S.$$

Let $\beta \in \operatorname{Aut} T$ and $X = (x_1, \ldots, x_m) \in S$. It may be easily checked that in $\operatorname{Aut} F \oplus T$, under the identifications made here, the element β acts diagonally on X, that is, $\beta^{-1} X \beta = (x_1 \beta, \ldots, x_m \beta)$. Also, $-I_f$, the inversion map on F, acts on S by inversion, that is, $(-I_f^{-1}) X(-I_f) = -X$. We may summarize this as follows.

LEMMA 2.2. *Suppose F is torsion-free and $T = \oplus_p T_p$ is finite. Suppose* $\operatorname{Aut} F \oplus T$ *is finite.*

(i) *Then* $\operatorname{Aut} F \oplus T$ *contains a normal subgroup*

$$H = \left(\bigoplus_p \left(\bigoplus_{i=1}^{d(p)} T_p\right)\right) \lhd (\operatorname{Aut} T \times \langle -I_F \rangle),$$

where $\operatorname{Aut} T$ *acts diagonally and $-I_F$ acts by inversion.*

(ii) $(\operatorname{Aut} F \oplus T)/H \simeq \operatorname{Aut} F/\langle -I_F \rangle$.

In particular, if $\operatorname{Aut} F \simeq C_2$, then the subgroup H is all of $\operatorname{Aut} F \oplus T$. The simplest case of this is when $F = C_\infty$, an infinite cyclic group, and $T = C_n$, a cyclic group of order n. For example, $\operatorname{Aut} C_\infty \oplus C_2 \simeq C_2 \oplus C_2$ and

$$\operatorname{Aut} C_\infty \oplus C_3 \simeq C_3 \lhd (C_2 \times C_2) \simeq D_3 \lhd C_2 \simeq D_6.$$

3. Abelian Automorphism Groups

Suppose that $\operatorname{Aut} A$ is a finite abelian group. Consider first the case in which $A = F$ is torsion-free. Then $\operatorname{Aut} A$ is a subdirect product of Hallett–Hirsch groups. Recall that G is a subdirect product of $H_1, \ldots, H_k$ if there is a monomorphism $\theta\colon G \to \Pi H_i$ such that θp_j is an epimorphism where $p_j\colon \Pi H_i \to H_j$ is the canonical projection. Hence, $\operatorname{Aut} A$ can be a subdirect product only of abelian groups. The only abelian Hallett–Hirsch groups are C_2, C_4, and C_6. $\operatorname{Aut} A$ must also satisfy the condition that if it contains an element of order 12, it must also contain an element of order 2 that is not a sixth power. Clearly, any such subdirect product is an abelian automorphism group. Thus, if A is torsion-free and $\operatorname{Aut} A$ is a finite abelian group, then $\operatorname{Aut} A$ is a direct product of cyclic groups of order 2, 3, or 4 such that $\operatorname{Aut} A$ is of even order and such that if $\operatorname{Aut} A$ has an element of order 12, then it has an element of order 2 that is not a sixth power. Also, any such finite abelian group occurs as $\operatorname{Aut} A$ for some torsion-free A.

Now suppose that $T \neq 0$, that is, suppose that A is not torsion-free. Since Aut A is abelian, Lemma 2.2 implies that Aut T acts trivially on T, that is, Aut $T = 1$. Hence, $T = C_2$. This means that Aut $A \simeq \mathrm{Hom}(F, C_2) \oplus \mathrm{Aut}\, F \simeq C_2^{(n)} \oplus \mathrm{Aut}\, F$. But these are precisely the same abelian automorphism groups that arise for A torsion-free.

4. Dihedral Automorphism Groups

We let D_n be the dihedral group of order $2n$ which may be generated by a and b subject to the relations

$$a^n = b^2 = 1,$$

$$b^{-1}ab = a^{-1}.$$

The only normal sbgroups of D_n are $\langle a^r \rangle$, $\langle a^2, ab \rangle$, $\langle a^2, b \rangle$, and D_n itself. Hence, the only homomorphic images of D_n are 1 and D_r where r divides n, remembering of course that $D_1 \simeq C_2$ and $D_2 \simeq C_2 \oplus C_2$. Now suppose that Aut $A = D_n$. From the sequence (2.1) we know that Aut $F \times$ Aut $T = H$ must be an image of D_n. We consider several cases.

First, suppose that $H = C_2$. Then Aut $T = 1$ since F has an automorphism of order 2. This forces T to be trivial or C_2. If T is trivial, then A is torsion-free and D_n is a subdirect product of Hallett–Hirsch groups. But in these groups elements of order 2 are central. Thus, the same is true for D_n. Therefore, $n = 1$ or 2. Suppose $T = C_2$. Then the sequence

$$\mathrm{Hom}(F, C_2) \rightarrowtail D_n \twoheadrightarrow C_2$$

splits and thus every element of D_n has order 1 or 2. Therefore, $n = 1$ or 2.

Suppose now that $H \simeq C_2 \oplus C_2$. Then Aut T is either trivial or C_2. The case when Aut T is trivial is discussed in the previous paragraph. If Aut $T = C_2$, then T is either C_3, C_4, or C_6. Suppose $T = C_3$. Then $\mathrm{Hom}(F, T) \simeq T$ since a normal abelian subgroup of D_n with an odd number of elements must be cyclic. Thus we have

$$C_3 \rightarrowtail D_n \twoheadrightarrow C_2 \times C_2$$

and D_n is of order 12 and $n = 6$. Suppose $T = C_4$ or C_6. Then the normal subgroup of D_n generated by $-I_A$ and $\mathrm{Hom}(F, T)$ is a non-cyclic normal abelian subgroup of order at least 8. But this is impossible.

Finally, suppose that Aut $F \times$ Aut $T \simeq D_m$ for $m \geqslant 3$. Then $\mathrm{Hom}(F, T)$ and hence T is cyclic. Then Aut T is abelian and hence lies in the center of D_m. However, $-I_F$ is non-trivial and central in D_m. But the center of D_m has at most two elements. Hence, Aut T is trivial. Thus, T is trivial or C_2. The former is impossible, while the latter implies that Aut F is an image of D_n. Hence, Aut $F = C_2 \oplus C_2$, but then the split exact sequence

$$C_2 \rightarrowtail D_n \twoheadrightarrow C_2 \oplus C_2$$

implies that D_n has no elements of order larger than 2. This contradiction completes the proof that Aut $A = D_n$ implies that $n = 1$, 2, or 6.

5. Quaternion Automorphisms Groups

We will now consider the case Aut $A = Q_n$, $n \geqslant 3$, the generalized quaternion group of order 2^n. If T has more than two elements, then Aut T has an element of order 2. The sequence

$$\operatorname{Hom}(F, T) \rightarrowtail Q_n \twoheadrightarrow \operatorname{Aut} F \times \operatorname{Aut} T$$

implies that Q_n has at least two elements of order 2, but this is impossible. If $T = C_2$, then Hom(F, T) has elements of order 2 which again is impossible. Hence, A is torsion-free. The Hallett–Hirsch classification now shows that $n = 3$, and that there is a torsion-free group A with Aut $A = Q_3$.

6. Central Cyclic-by-Simple Automorphism Groups

Suppose now that Aut $A = G$ where G is finite, G/ZG is a non-abelian simple group, and ZG is cyclic. Since Hom(F, T) is normal, it follows that Hom$(F, T) \leqslant ZG$. Similarly, $-I_t$ lies in ZG. It follows that the group generated by $-I_t$ and Hom(F, T) is contained in the cyclic group ZG and splits over Hom(F, T). Hence, Hom(F, T), and consequently, T, is trivial. Therefore, A is torsion-free and by the Hallett–Hirsch classification Aut A is solvable. This contradiction proves (v) of Theorem 1.2.

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